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Stability of vector bundles from F-theory

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Abstract: We use a recently proposed construction of stable holomorphic vector bundles V on elliptically fibered Calabi-Yau n-fold Z_n in terms of F-theory compactifications on local singularities to describe stability conditions on V. Specifically, the requirement that the F-theory compactification manifold is Calabi-Yau implies a a stability criterion on V which is formulated in terms of the existence of holomorphic sections of certain line bundles.

Keywords: Superstrings and Heterotic Strings, F-Theory, String Duality, Superstring Vacua.

The heterotic string compactified on Calabi-Yau n fold Z_n gives rise to interesting minimal supersymmetric string vacua with a quite realistic spectrum of gauge symmetries and matter fields [1]. The definition of these vacua, involves the specification of a gauge background V with structure group $H \in G_0$ embedded in the perturbative heterotic gauge group $G_0 = E_8 \times E_8$ or SO(32) such that

$$c_1(V) = 0 \bmod 2\,, \qquad \lambda(V) = c_2(Z) + [W]\,, \tag{1}$$

where λ is the four-dimensional characteristic class of V. The second term $[W]$ applies to a generalization of heterotic vacua which include non-perturbative five-branes [2]. Here $[W]$ is a sum of formal four-forms that integrate to one in the transverse direction to the five-branes. Note that the five-branes necessarily have to wrap two-cycles in a 3-fold compactification; such cycles have to be holomorphic in order to preserve supersymmetry [3].

In addition, the connection on V has to satisfy

$$F_{ab} = F_{\bar{a}\bar{b}} = 0\,, \qquad g_{a\bar{b}}F^{a\bar{b}} = 0\,. \tag{2}$$

The first equation says that V is holomorphic, while the second equation is interpreted as some stability condition on V. Although an existence theorem exists under certain conditions [4], explicit solutions to (2) have been known only in very special constructions. The difficulty to find appropriate backgrounds V has hampered the study of interesting four-dimensional vacua for a long time.

An entirely independent construction of holomorphic, stable vector bundles[1] V on Calabi-Yau n-folds Z_n has been developed in [6] using type-IIA strings compactified on local Calabi-Yau $n+1$ fold singularities $\mathcal{W}_{n+1}$ and the F-theory limit thereof.[2] This geometric approach gives a systematic way to obtain families of vector bundles that solve (2), with any structure group $H \in G_0$ and on any toric Calabi-Yau manifold Z. Given the fact that it is so hard to find solutions to (2) in general, there must be a very good reason why the geometric construction yields a large class of solutions without difficulties. Now supersymmetry of the F-theory compactification requires that the singularity $\mathcal{W}_{n+1}$ is Calabi-Yau. It is the purpose of this note to show that it is precisely this Calabi-Yau condition that translates to a stability condition on the bundle V on Z, formulated in terms of the existence of holomorphic sections of certain line bundles. The answer to why it was so easy to get solutions to (2) from the geometric construction is thus that we have replaced the complicated condition (2) of the heterotic string, about which we have poor control, with the simple Calabi-Yau condition on the type-IIA/F-theory side, which is much easier to deal with.

[1]We will loosely call V a bundle in the following, though our construction naturally yields sheaf generalizations of bundles, which provide the relevant set up for heterotic string compactifications [5].

[2]See [7] for a review.

The relation of type-IIA strings and holomorphic stable vector bundles can be traced back to a simple extension of mirror symmetry in a $K3 \times T^2$ compactification [8], more specifically a discrete symmetry in the moduli space of this compactification that relates geometric deformations of K3 to Wilson lines on the T^2 factor. One can then combine this picture with taking the F-theory limit of the type-IIA compactification. Specifically one can show that the moduli of Wilson lines on $Z_1 = T^2$ is the same as the geometric moduli of a local 2-fold singularity $\mathcal{W}_2$. The relevant compactification geometry $\mathcal{W}_2$ is the local mirror of a Kähler resolution of an H singularity of an elliptically fibered ALE space. Here H is the structure group of the bundle V, as before.

Fibering this picture, using the adiabatic argument of [9], one obtains an equivalence between complex deformations of an elliptic and K3 fibered Calabi-Yau singularity $\mathcal{W}_{n+1}$ and holomorphic stable vector bundles on an elliptically fibered manifold Z_n [6]. Holomorphic stable bundles V on smooth elliptic fibrations Z_n have been first considered in [10] and subsequently in [11].

The construction in [6] has been formulated in terms of toric geometry which for many purposes is the most useful and most general definition for the Calabi-Yau manifolds $\mathcal{W}_{n+1}$ and Z_n. The toric construction of the pair (V, Z_n) in terms of an $n+1$ dimensional non-compact Calabi-Yau manifold $\mathcal{W}_{n+1}$ works for any structure group H of the bundle. If H fits into the perturbative heterotic gauge group $G_0 = E_8 \times E_8$ or SO(32), we can interpret (V, Z_n) as a valid perturbative heterotic vacuum. Precisely in this case one can find an embedding of the geometry $\mathcal{W}_{n+1}$ into a *compact* Calabi-Yau manifold W_{n+1}. The statement that F-theory compactified on the local patch $\mathcal{W}_{n+1}$ of W_{n+1} is equivalent to the heterotic string compactification on (V, Z_n) then establishes F-theory/heterotic duality [12, 13] in the point particle limit.

Toric geometry describes a Calabi-Yau n-fold Z_n in terms of a convex, toric polyhedron $\Delta^*_{W_{n+1}}$, which is the convex hull of a set of integral vertices $\nu^\star_i$ in a standard integral lattice N.[3] It is satisfying to observe [6] that the heterotic physics such as the compactification manifold Z_n and some defining data of the bundle V have a very simple representation in terms of the polyhedron $\Delta^*_{W_{n+1}}$. Moreover also non-perturbative dynamics of the heterotic vacuum for singular configurations, in particular non-perturbative gauge symmetries and non-perturbative five-branes that appear in (1), can be read of from the toric data of W_{n+1} in a simple way. It is the purpose of this note to extend the dictionary between physical and toric data developed in [6] to describe stability of the bundle V in terms of convexity of the polyhedron $\Delta^*_{W_{n+1}}$.

In particular it was noted in [6], that for a given structure group H of the bundle, convexity of the toric polyhedron $\Delta^\star_{\mathcal{W}_{n+1}}$ translates to a bound on the first Chern class $\eta = c_1(\mathcal{N})$ of a line bundle $\mathcal{N}$ that is an important characteristic of the bundle V.

[3]See [6] for a list of reviews on toric geometry in the physics literature.

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Observing that in six dimensions, the bound on η from convexity of $\Delta^\star$ translates to the stability of instantons on K3 with structure group H, these bounds were interpreted as a similar stability condition in lower dimensions. A general formula for the stability of V in terms of bounds on η was subsequently conjectured in ref. [14] which was however not in agreement[4] with the original bounds in [6]. In the following we derive a general stability bound for η from the toric representation of data (V, Z_n) which is consistent and, we believe, gives a correct treatment.

We start with a sketch of the description of the pair (V, Z_n) in terms of the non-compact Calabi-Yau $n+1$-fold $\mathcal{W}_{n+1}$. The latter is elliptically fibered with base $\tilde{B}_n$ and ALE fibered with base B_{n-1}. The manifold $\mathcal{W}_{n+1}$ is defined as the vanishing locus of a polynomial p in a toric ambient space with coordinates $(y, x, \tilde{z}, v, x_i)$, where x_i denote the coordinates of the base B_{n-1} and $(y, x, \tilde{z}, v)$ are certain coordinates on the ALE fiber. The general form of p is

$$p = p_0 + p_+ = p_0(y, x, \tilde{z}; x_i) + \sum_{j=1}^{J} v^j p_+^j(y, x, \tilde{z}; x_i)\,, \tag{3}$$

where p_0 and p_+^j are quasi-homogeneous polynomials. In particular the v independent piece p_0 describes an elliptically fibered Calabi-Yau manifold $Z_n:\ p_0 = 0$, while the polynomials p_+^j contain the information about the bundle V. More specifically, each node in the affine Dynkin diagram $\Gamma(H)$ with Dynkin index s_i contributes one monomial to the polynomial $p_+^{s_i}$; in particular $J = \max(s_i)$. E.g. in the case $H = \mathrm{SU}(N)$, we have N nodes of index 1, each of which contributes a monomial to p_+^1. The precise form of the v dependent part of p is $p_+ = v(a_N(x_i)\tilde{z}^N + a_{N-2}(x_i)\tilde{z}^{N-2}x + \cdots + a_0(x_i)x^{N/2})$ for N even with the last term being $\sim yx^{(N-3)/2}$ for N odd. The complex parameters multiplying the monomials in p_+ give a projective parametrization of the moduli space of the $\mathrm{SU}(N)$ bundle.

Thus the structure group H determines the $y, x, \tilde{z}, v$ content of the monomials appearing in the defining equation p of $\mathcal{W}_{n+1}$. It does not determine the dependence on the base variables x_i, however. The latter specifies topological properties of V that enter the higher Chern classes $c_k(V),\ k > 1$. However note that the dependence on x_i is quite restricted because of the quasi-homogeneousness of p. It was shown in [6] that the dependence of the monomials in p on x_i is completely determined by specifying the action of two C^* actions acting on $(y, x, \tilde{z}, v, x_i)$. In other words, we have to specify two line bundles $\mathcal{L}$ and $\mathcal{M}$ on the base B_{n-1} and the coordinates $(y, x, \tilde{z}, v, x_i)$ transform as certain sections of $\mathcal{M}$ and $\mathcal{L}$ such that

$$y \sim \mathcal{M}^3\mathcal{L}^3\,, \qquad x \sim \mathcal{M}^2\mathcal{L}^2\,, \qquad \tilde{z} \sim \mathcal{M}\,, \qquad v \sim \mathcal{M}^{5-N}\,, \qquad f_{c,d} \sim \mathcal{M}^d\mathcal{L}^c\,, \tag{4}$$

[4]Essentially, the discriminant criterion used in [14] is not restrictive enough to fix the structure group H.

where $f_{c,d} = f_{c,d}(x_i)$ is the base dependence part in the monomial

$$y^{(6-2b-c)/3} x^b \tilde{z}^c v^d f_{c,d}(x_i)$$

of p. Moreover N is the highest power of $\tilde{z}$ that appears in p_+^1.

From the fact that $p_0 = 0$ describes a Calabi-Yau manifold it follows that the bundle $\mathcal{L}$ is actually the anti-canonical bundle of B_{n-1}. On the other hand $\mathcal{M}$ is an important intrinsic property of V. In the following we will derive a bound on the first Chern class $c_1(\mathcal{N})$ of the line bundle $\mathcal{N} = \mathcal{M}\mathcal{L}^6$ from convexity of the polyhedron $\Delta^*_{W_{n+1}}$, which is the toric way to ensure that the singularity $\mathcal{W}_{n+1}$ represents a patch of a Calabi-Yau space.

The change from $\mathcal{M}$ to $\mathcal{N}$ is intended to make contact with the notation of ref. [10], where V has been described in terms of sections of a certain weighted projective bundle $\mathcal{W}$ on B_{n-1} with coordinates on $\mathcal{W}$ being sections of $\mathcal{N}^{s_i}\mathcal{L}^{-d_j}$. Here s_i are the Dynkin indices as above and d_i are the degrees of the independent Casimir invariants of H. The dependence of higher Chern classes of V on $c_1(\mathcal{N})$ has been determined in [10, 15].

To keep the discussion as basic as possible we will work directly with the defining polynomial p rather than with the toric polyhedron $\Delta^{\star}_{\mathcal{W}_{n+1}}$ that determines p [16]. In particular, if we consider a structure group $H \in G_0$, we have a compact embedding $\mathcal{W}_{n+1} \to W_{n+1}$. Moreover the K3 fiber (that, in a local patch, contains the ALE fiber of $\mathcal{W}_{n+1}$) has a singularity G which is the commutant of H in G_0 [13]. The defining polynomial $\hat{p}$ of W_{n+1} can be written in generalized Weierstrass form:

$$\hat{p} = y^2 + x^3 + yx\hat{z}a_1 + x^2\hat{z}^2a_2 + y\hat{z}^3a_3 + x\hat{z}^4a_4 + \hat{z}^6a_6\,. \tag{5}$$

Note that we use $(y, x, \hat{z})$ and $(y, x, \tilde{z})$ to denote the homogeneous coordinates of the elliptic fiber of the $n+1$-dimensional Calabi-Yau W_{n+1} and the n dimensional Calabi-Yau Z_n, respectively. The a_n are functions of the coordinates $\tilde{x}_i$ of the base $\tilde{B}_n$ of the *elliptic* fibration of W_{n+1}. In particular having a singularity of type G above a locus, say at $z \equiv \tilde{x}_1 = 0$ on the base $\tilde{B}_n$ can be phrased in terms of the behavior of the a_n near $z = 0$, $a_n \sim z^{\delta_n}$ using Tate's algorithm [17].

To reiterate, the structure group H determines the singularity G of the K3 fiber of W_{n+1}. In turn G is determined by the behavior of the coefficient functions a_n in the generalized Weierstrass form $\hat{p}$ near the singularity $z = 0$. Moreover we are interested in the singularity of the K3 fiber W_2 of the K3 fibration $W_2 \to W_{n+1} \to B_{n-1}$. Thus z is now a coordinate on the base $\mathbb{P}^1$ of the elliptically fiberered K3, $\pi : W_2 \to \mathbb{P}^1$. However note that in the local limit $W_{n+1} \to \mathcal{W}_{n+1}$ where our map $f : \mathcal{W}_{n+1} \to (V, Z_n)$ applies, the a_n become nothing but the $\tilde{B}_n$ dependent parts of the monomials $y^{(6-b-c)/3}x^b\tilde{z}^cv^df_{c,d}(x_i)$ in the defining polynomial p of $\mathcal{W}_{n+1}$.

To be more explicit let us consider the case $G_0 = E_8 \times E_8$. The relation between the coordinates in $\hat{p}$ and p, using $z \equiv \tilde{x}_1,\ w \equiv \tilde{x}_2$ as the coordinates of the base $\mathbb{P}^1$

of W_2 as above, is then

$$(y, x, \tilde{z}; v; x_i)_p = \left(y, x, \hat{z}zw; \frac{w}{z}; \tilde{x}_i,\ i > 3\right)_{\hat{p}} .$$

In particular the existence of a term $a_n = z^{\delta_n} f(x_i)$ implies that the line bundle

$$\mathcal{N}^{d_n} \mathcal{L}^{n-6d_n} \tag{6}$$

has a holomophic section, where we have defined $d_n = n - \delta_n$. Combining the conditions imposed by all the a_n, $n \in \{1, 2, 3, 4, 6\}$ we arrive at the condition

$$\nu(G)\, c_1(\mathcal{L}) \leq\ \eta \qquad (\leq 12 c_1(\mathcal{L}))\, , \tag{7}$$

where $\nu(G)$ is the singularity dependent quantity

$$\nu(G) = \max\left(\frac{(6d_n - n)}{d_n}\right), \tag{8}$$

for all $n \in \{1, 2, 3, 4, 6\}$ with $d_n \neq 0$. Note that this argument also tells us that the minimal bundle $\mathcal{N}_{min}$ is always a power of $\mathcal{L}$, a fact that is non-trivial for $h^{1,1}(B) > 1$. We have also indicated in (7) an upper bound for η of a very different origin. It is not related to the stability of $\mathcal{N}$ but rather to the stability of $\eta' = c_1(\mathcal{N}') \geq 0$, where $\mathcal{N}'$ is the equivalent bundle in the second E_8 factor.

From eqs. (7) and (8) we obtain

$$\begin{array}{ll}
G = \mathrm{SU}(2)^* : \eta \geq \dfrac{14}{3} \cdot L & G = E_8 : \eta \geq 0 \cdot L \\
G = \mathrm{SU}(3)^* : \eta \geq \dfrac{9}{2} \cdot L & G = E_7 : \eta \geq 2 \cdot L \\
G = \mathrm{SU}(M) : \eta \geq 5 \cdot L & G = E_6 : \eta \geq 3 \cdot L \\
G = \mathrm{SO}(N) : \eta \geq 4 \cdot L & G = G_2 : \eta \geq 4 \cdot L \\
 & G = F_4 : \eta \geq 3 \cdot L
\end{array} \tag{9}$$

Here $N \in \{7, \ldots, 12\}$, $M \in \{2, \ldots, 6\}$ and the groups with a star denote the singularities associated to III and IV^s fibers in the list of [17]. Moreover $L = c_1(\mathcal{L})$. Note that our arguments for the bound (7) give a necessary but in general not sufficient criterion for the stability of the bundle. In particular the distinction between singularities associated to simply laced (SL) and non-simply laced (NSL) groups is often the specific form of the sections $f_{c,d}$ in the SL case rather than a difference in the topological class [17]. Our arguments give a criterion for the existence of a holomorphic section $f_{c,d}$ which is sufficient for SL groups, but it does not ensure that one gets a generic enough section to describe also the NSL case. Thus the bounds given in (9) are strict lower bounds only for the simply laced case.

It would be nice to have a derivation of the bound on η that is directly related to the structure group H and in particular also applies to the non-compact case with arbitrary rank. A reasonable proposal based on (9) is

$$\begin{aligned}
&H = \mathrm{SU}(N) : \eta \geq N \cdot L && H = E_8 : \eta \geq 5 \cdot L \\
&H = \mathrm{SO}(7) : \eta \geq 4 \cdot L && H = E_7 : \eta \geq \frac{14}{3} \cdot L \\
&H = \mathrm{SO}(M) : \eta \geq \frac{M}{2} \cdot L && H = E_6 : \eta \geq \frac{9}{2} \cdot L \\
&H = \mathrm{Sp}(K) : \eta \geq 2K \cdot L && H = G_2 : \eta \geq \frac{7}{2} \cdot L \\
& && H = F_4 : \eta \geq \frac{13}{3} \cdot L \,,
\end{aligned} \tag{10}$$

where $N \geq 2$, $M \geq 8$ and $K \geq 2$. Here we have used the calculation of the six-dimensional spectrum in [17, table 3] to raise the bounds for the NSL cases, such that we get the correct answer in six dimensions. Similarly we could compare the calculation of the matter spectrum in four dimension for a heterotic compactification on a Calabi-Yau 3-fold with (9).

It is interesting to observe, how the geometric description contains non-trivial information about the instanton dynamics on Calabi-Yau manifolds Z_n. Moreover in many cases there is also another point of view, namely in terms of a supersymmetric field theory in uncompactified space time. In particular, via the heterotic compactification on $K3 \times T^2$, the instanton dynamics on K3 is related to the moduli space of $\mathcal{N} = 2$ supersymmetric field theories. Similarly, instantons on Calabi-Yau three-folds will be related to the moduli space of $\mathcal{N} = 1$ Super-Yang-Mills theories. It would be very interesting to use the toric construction of V on Z_n to study this relation in detail.

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References

[1] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, *Nucl. Phys.* **B 258** (1985) 46;
C.M. Hull and E. Witten, *Supersymmetric sigma models and the heterotic string*, *Phys. Lett.* **B 160** (1985) 398;
E. Witten, *New issues in manifolds of* SU(3) *holonomy*, *Nucl. Phys.* **B 268** (1986) 79;

J. Distler, *Resurrecting* $(2,0)$ *compactifications*, *Phys. Lett.* **B 188** (1987) 431;
J. Distler and B. Greene, *Aspects of* $(2,0)$ *string compactifications*, *Nucl. Phys.* **B 304** (1988) 1.

[2] M.J. Duff, R. Minasian and E. Witten, *Evidence for heterotic/heterotic duality*, *Nucl. Phys.* **B 465** (1996) 413 [hep-th/9601036].

[3] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158];
M. Bershadsky, C. Vafa and V. Sadov, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9511222].

[4] K. Uhlenbeck and S.T. Yau, *On the existence of hermitian Yang-Mills connections in stable vector bundles*, preprint 1986.

[5] J. Distler, B.R. Greene and D.R. Morrison, *Resolving singularities in* $(0,2)$ *models*, *Nucl. Phys.* **B 481** (1996) 289 [hep-th/9605222].

[6] P. Berglund and P. Mayr, *Heterotic string/F-theory duality from mirror symmetry*, *Adv. Theor. Math. Phys.* **2** (1999) 1307 [hep-th/9811217].

[7] P. Mayr, $N=1$ *heterotic string vacua from mirror symmetry*, hep-th/9904115, to appear in the proceedings of the the Winter School on *Vector bundles, mirror symmetry, and lagrangean submanifolds*, Harvard University, January 1999.

[8] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of 4D* $N=2$ *gauge theories, I*, *Adv. Theor. Math. Phys.* **1** (1998) 53 [hep-th/9706110].

[9] C. Vafa and E. Witten, *Dual string pairs with* $N=1$ *and* $N=2$ *supersymmetry in four-dimensions* *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 225 [hep-th/9507050]

[10] R. Friedman, J. Morgan and E. Witten, *Vector bundles and F-theory*, *Comm. Math. Phys.* **187** (1997) 679 [hep-th/9701162].

[11] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, *On four-dimensional compactifications of F-theory*, *Nucl. Phys.* **B 505** (1997) 165 [hep-th/9701165];
R.Y. Donagi, *Asian J. Math.* **1** (1997) 214; *MSRI pub.* **28** (1992) 65;
G. Curio and R.Y. Donagi, *Moduli in n=1 heterotic/f theory duality*, *Nucl. Phys.* **B 518** (1998) 603 [hep-th/9801057].

[12] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[13] D.R. Morrison and C. Vafa, *Compactifications of F-theory on Calabi-Yau threefolds, I*, *Nucl. Phys.* **B 473** (1996) 74 [hep-th/9602114]; *Compactifications of F-theory on Calabi-Yau threefolds, II* *Nucl. Phys.* **B 476** (1996) 437 [hep-th/9603161].

[14] G. Rajesh, *Toric geometry and F-theory/heterotic duality in four dimensions*, *J. High Energy Phys.* **12** (1998) 018 [hep-th/9811240].

[15] B. Andreas, *On vector bundles and chiral matter in $N = 1$ heterotic compactifications*, *J. High Energy Phys.* **01** (1999) 011 [hep-th/9802202];
G. Curio, *Chiral matter and transitions in heterotic string models*, *Phys. Lett.* **B 435** (1998) 39 [hep-th/9803224].

[16] V. Batyrev, *Duke Math. Journ.* **69** (1993) 349.

[17] M. Bershadsky et al., *Geometric singularities and enhanced gauge symmetries*, *Nucl. Phys.* **B 481** (1996) 215 [hep-th/9605200].